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STRENGTH CRITERIA OF AN ANISOTROPIC MATERIAL*

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Strength criteria are proposed for anisotropic materials as a generalization of the well-known phenomenological criteria for an isotropic medium based on the introduction of certain functions of the stress tensor invariants.

1. The viewpoint, according to which a composite is treated as a certain reduced homogeneous body /1, 2/, is well-known. If even each component of the composite is isotropic here, the reduced body possesses an anisotropy which is customarily called structural /2/.

A fairly large number of strength criteria, that agree to some extend with experimental data /3, 4/, have been developed for isotropic materials. The majority are based on the introduction of a certain function, which depends on the stress tensor, that describes a surface encompassing the safe stress states in the stress space

$$F(Y_1, Y_2, Y_3) = 0 \tag{1.1}$$

The function (1.1) should understandably depend on the temperature and possibly other parameters of a physicochemical nature. However, for simplicity we shall consider all these parameters to be fixed. Here Y_α ($\alpha = 1, 2, 3$) are three independent invariants of a symmetric stress tensor /5/, for which we can select, say

$$Y_1 = \Theta = \sigma_{ii},\ Y_2 = \sigma_u = (s_{ij}s_{ij})^{1/2},\ Y_3 = \det|s_{ij}| \tag{1.2}$$

where σ_u is the intensity of the stress tensor $\|\sigma_{ij}\|$; summation from 1-3 is over repeated subscripts.

It is sometimes assumed that the function F is independent of the third invariant Y_3 and the criterion (1.1) is represented in the form

$$f(\sigma_u) = K(\Theta) \tag{1.3}$$

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where K is a certain material "constant" which depends on the hydrostatic pressure Θ. In the simplest case, the function f is assumed to be linear so that the criterion (1.3) takes the form

$$\sigma_u = K(\Theta) \tag{1.4}$$

The so-called tensor-polynomial formulation (2, 6-8/

$$(F_{ij}^{(0)}\sigma_{ij})^{\alpha_0} + (F_{ijkl}\sigma_{ij}\sigma_{kl})^{\alpha_1} + (F_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn})^{\alpha_2} + \ldots = 1 \tag{1.5}$$

is sometimes used in considering the phenomenological fracture criteria of anisotropic media, where $F^{(q)}$ are tensors of rank $2(q+1)$, called the strength tensors, and α_l are certain constants $(q = 0, 1, \ldots)$. **Most often it is assumed that** $q < 2$ in (1.5). Then in the case of an isotropic medium, not only the criterion (1.1) but even the criteria (1.3) and (1.4) follow from the representation (1.5).

New strength criteria are proposed below for anisotropic materials as a generalization of the criteria (1.1), (1.3) and (1.4).

2. We consider first a transversally-isotropic material (TIM). Let its axis of transversal isotropy be characterized by a vector with the components c_i in a certain rectangular Cartesian coordinate system.

The symmetric stress tensor for the TIM has five independent invariants /5/, two of which are linear and two quadratic /9/

$$\sigma = c_i\sigma_{ij}c_j,\ \bar{\sigma} = {}^1/_2\sigma_{ij}(\delta_{ij} - c_ic_j)$$
$$Q = (Y - \sigma^2)^{1/2},\ P = [\sigma_{ij}\sigma_{ij} + \sigma^2 - 2(Y + \bar{\sigma}^2)]^{1/2}$$
$$(Y \equiv c_i\sigma_{ik}\sigma_{kj}c_j)$$

Then the criteria for TIM that correspond to criterion (1.1) for an isotropic medium can be written in the form

$$F(\sigma,\ \bar{\sigma},\ P,\ Q,\ Y_3) = 0 \tag{2.1}$$

where Y_3 is determined in (1.2), for example. The identities

$$\Theta \equiv \sigma + 2\bar{\sigma},\ \sigma_u \equiv [P^2 + 2Q^2 + {}^2/_3(\sigma - \bar{\sigma})^2]^{1/2}$$

are obvious.

Consequently, the criteria (1.3) and (1.4) for TIM can be generalized in the form

$$f(P,\ Q,\ |\sigma - \bar{\sigma}|) = K(\sigma,\ \bar{\sigma}) \tag{2.2}$$
$$P^2 + \alpha_1Q^2 + \alpha_2(\sigma - \bar{\sigma})^2 = K^2(\sigma,\ \bar{\sigma})$$
$$P^2 + 2Q^2 + 2/3(\sigma - \bar{\sigma})^2 = K^2(\sigma,\ \bar{\sigma})$$

Here and henceforth $\alpha_1, \ldots$ are certain constants.

The criterion (2.2) can, of course be represented in the form

$$f(P,\ Q) = K(\sigma,\ \bar{\sigma}),\quad P^2 + \alpha_1Q^2 = K^2(\sigma,\ \bar{\sigma}),\quad P^2 + 2Q^2 = K^2(\sigma,\ \bar{\sigma}) \tag{2.3}$$

However, it is more difficult to see the relationship with criteria (1.3) and (1.4) for the isotropic case from (2.3).

3. For orthotropic materials the symmetric stress tensor has six independent invariants $P_1, \ldots, P_6$ /9/ (as in the case of the most general anisotropy), where three are linear.

Let the principal axes or orthotropy be characterized by an orthonormalized reference point with the components $c_i^{(\varkappa)}$ $(\varkappa,\ i = 1, 2, 3)$. It is convenient to take as independent invariants

$$P_\varkappa = c_i^{(\varkappa)}\sigma_{ij}c_j^{(\varkappa)},\ P_{\varkappa+3} = (V_iV_i - 2V_\varkappa^2)^{1/2}$$
$$(V_\varkappa \equiv c_i^{(\varkappa)}\sigma_{ik}\sigma_{kj}c_j^{(\varkappa)},\ \varkappa = 1, 2, 3)$$

The identities

$$\Theta \equiv P_1 + P_2 + P_3,\ \sigma_u \equiv {}^1/_3[(P_1 - P_2)^2 + (P_2 - P_3)^2 +$$
$$(P_3 - P_1)^2] + P_4^2 + P_5^2 + P_6^2$$

are easily established.

Then the criterion (1.1) for an orthotropic medium can be generalized in the form

$$F(P_1,\ \ldots,\ P_6) = 0 \tag{3.1}$$

and the criteria (1.3) and (1.4) in the form

$$f(P_4,\ P_5,\ P_6,\ |P_1 - P_2|,\ |P_2 - P_3|,\ |P_3 - P_1|) = \tag{3.2}$$
$$K^2(P_1,\ P_2,\ P_3),\ P_4^2 + \alpha_1P_5^2 + \alpha_2P_6^2 +$$

$$\alpha_3[(P_1 - P_2)^2 + (P_2 - P_3)^2 + (P_3 - P_1)^2] =$$
$$K^2(P_1, P_2, P_3), \quad P_4^2 + P_5^2 + P_6^2 +$$
$$^1/_3 [(P_1 - P_2)^2 + (P_2 - P_3)^2 + (P_3 - P_1)^2 = K^2(P_1, P_2, P_3)$$

In this case also the criteria (3.2) can be represented in the form

$$f(P_4, P_5, P_6) = K(P_1, P_2, P_3), \quad P_4^2 + \alpha_1 P_5^2 + \alpha_2 P_6^2 = K^2(P_1, P_2, P_3), \quad P_4^2 + P_5^2 + P_6^2 = K^2(P_1, P_2, P_3) \tag{3.3}$$

although it is more difficult to see the relationship with the criteria (1.3) and (1.4) for an isotropic medium for (3.3).

We also note that the criteria (3.1)-(3.3) are more general than their corresponding criteria (2.1)-(2.3): no assumption on the absence of the cubic invariant (1.2) is required for their formulation because the arguments of the functions F in (3.1) are only linear and quadratic invariants.

4. Analogous criteria can be formulated for media with arbitrary anisotropy. For the medium under consideration let the stress tensor have N independent invariants $Q_1, \ldots, Q_N$, where m are linear, and $(n - m)$ are quadratic $(m \leqslant 3)$. Here the quantities Q_i $(i = 1, \ldots, N)$ are understood to be the joint invariants of the stress tensor with the tensors giving the geometric symmetry of the medium. Such tensors are written down for all textures and syngonies in /10/, from which it also follows that N is finite. Then the generalization of the criterion (1.1) will be

$$F(Q_1, \ldots, Q_N) = 0 \tag{4.1}$$

We introduce additionally a quasilinearity postulate /9/ which, as it applies to (4.1), is that F depends only on linear and quadratic invariants, where $N \leqslant 6$.

After this, the generalization of the criterion (1.3) can be formulated as

$$f(Q_{m+1}, \ldots, Q_n) = K(Q_1, \ldots Q_m)$$

where $m \leqslant 3$. Taking account of the identity

$$\Theta \equiv \sum_{\varkappa=1}^{m} a_\varkappa Q_\varkappa, \quad \sigma_u^2 \equiv \sum_{\gamma=m+1}^{n} Q_\gamma^2 + \sum_{\varkappa \neq \rho = 1}^{m} (a_\varkappa Q_\rho - a_\rho Q_\varkappa), \quad n \leqslant 6$$

the generalization of the criteria (1.3) and (1.4) can be formulated in the form

$$f(Q_{m+1}, \ldots, Q_n, \ |a_1 Q_2 - a_2 Q_1|, \ldots, |a_m Q_1 - a_1 Q_m|) = K(Q_1, \ldots, Q_m)$$
$$\sum_{\gamma=m+1}^{n} \alpha_\gamma Q_\gamma^2 + \sum_{\varkappa \neq \rho = 1}^{m} (a_\varkappa Q_\rho - a_\rho Q_\varkappa)^2 = K^2(Q_1, \ldots, Q_m)$$
$$\sum_{\gamma=m+1}^{n} Q_\gamma^2 + \sum_{\varkappa \neq \rho = 1}^{m} (a_\varkappa Q_\rho - a_\rho Q_\varkappa)^2 = K^2(Q_1, \ldots, Q_m)$$

Here $a_\varkappa$ are numbers subject to the condition (1.5)

$$\sum_{\varkappa=1}^{m} a_\varkappa^2 = 3 \tag{4.2}$$

In particular, for an isotropic medium $m = 1$ $(n = 2)$, and it consequently follows from (4.2) that

$$a_1 = \sqrt{3}, \ Q_1 = {}^1/_3 \sqrt{3}\Theta, \ Q_2 = \sigma_u$$

For a transversally-isotropic medium $m = 2$ $(n = 4)$, and in this case /11/

$$a_1 = 1, \ a_2 = \sqrt{2}, \ Q_1 = \sigma, \ Q_2 = \sqrt{2}\bar{\sigma}, \ Q_3 = P, \ Q_4 = \sqrt{2}Q$$

As already mentioned, for an orthotropic medium $m = 3$ $(n = 6)$; in this case

$$a_1 = a_2 = a_3 = 1, \ Q_i = P_i \ (i = 1, \ldots, 6)$$

Of course, all the strength criteria proposed above require experimental confirmation even though their sole advantage over already existing criteria is the generality and logical connection with corresponding **criteria for isotropic materials.**

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THE SPATIAL PROBLEM OF THE COMPRESSION OF A MATERIAL ALONG A PERIODIC SYSTEM OF PARALLEL CIRCULAR CRACKS*

V.M. NAZARENKO

The non-axisymmetric problem of the biaxial uniform compression of a material along a periodic system of parallel circular cracks is considered. A facture criterion is used /1, 2/ within the framework of linearized stability theory according to which the beginning of fracture of the material under compression along the cracks is characterized by local buckling near the cracks. Within the framework of this approach, axisymmetric and plane problems were considered earlier for different material models (highly-elastic, composite and plastic) for one or two internal cracks, near-surface cracks and a periodic system of cracks /1-13/**. (**See also: Nazarenko, V.M., The axisymmetric problem of the fracture mechanics of materials under compression along a periodic system of parallel cracks (unequal roots).

Proceeding of the Eleventh Scientic Conf. of Young Scientists. Inst. Mechanics, Ukraine Academy of Sciences, Kiev, 1986. 154-161, Dep. VINITI 5507-86, July 28, 1986 Nazarenko, V.M. and Starodubtsev, I.P., On material fracture under compression along two parallel cracks in the case of plane strain. Non-classical and Mixed Problems of the Mechanics of a Deformable Body: Materials of a Seminar of Young Scientists, Kiev, 1985, 142-145, Dep. 5531-85 in VINITI, July 29, 1985.) The investigation is performed in general form for an arbitrary kind of elastic potential for compressible and incompressible materials, the theory of large and modifications of small subcritical strains, and can be extended to other models of a deformable body (composites, plastic bodies, etc.).

1. Formulation of the problem. Fracture of a material weakened by a periodic system of parallel disc-shaped coaxial cracks $\{r < a, 0 \leqslant \theta < 2\pi, x_3 = 2hn, n = 0, \pm 1, \pm 2, \ldots\}$ under biaxial compression in planes parallel to the cracks is considered. Lagrange coordinates x_j $(j = 1, 2, 3)$ are utilized that are identical with the Cartesian coordinates in the undeformed state, as are the symmetric stress tensor S^0 referred to unit area of the body in the undeformed state, $\mathbf{u}$, $\mathbf{t}$ is the perturbation of the displacement vector and the non-symmetric Kirchhoff stress tensor, respectively, and r, θ, x_3 are the cylindrical coordinates obtainable from the Cartesian coordinates x_j $(j = 1, 2, 3)$.

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